Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

As to $\phi(a+bx+cx^2+\ldots)$, let $b+cx+dx^2+\ldots$ be z: then

$$\phi(a+xz)=\phi a+\phi' a.xz+\frac{\phi'' a}{2}x^2z^2+\frac{\phi''' a}{2.3}x^3z^3+\ldots.$$

$$=\phi a+\phi' a.x(b+Db.x+D^2b.x^2+\ldots)+\frac{\phi'' a}{2}x^2(b^2+Db^2.x+D^2b.x^2+\ldots)$$

$$+\frac{\phi''' a}{2.3}x^3(b^3+Db^3.x+D^2b^3.x^2+\ldots)+\ldots.$$

$$=\phi a+\phi' a.bx+\left(\phi' a.Db+\frac{\phi'' a}{2}\cdot b^2\right)x^2$$

$$+\left(\phi' a.D^2b+\frac{\phi'' a}{2}\cdot Db^2+\frac{\phi''' a}{2.3}\cdot b^3\right)x^3$$

$$+\left(\phi' a.D^3b+\frac{\phi'' a}{2}\cdot D^2b^2+\frac{\phi''' a}{2.3}Db^3+\frac{\phi^{\text{IV}} a}{2.3.4}b^4\right)x^4+\ldots$$

$$=\phi a+D\phi a.x+D^2\phi a.x^2+D^3\phi a.x^3+D^4\phi a.x^4+\ldots.$$

by the mode of forming $D^4\phi a$ already shown.

I shall be happy to solve any difficulties which any of your readers may find.

On a Table for the Formation of Logarithms and Anti-Logarithms to Twelve Places. By PETER GRAY, *F.R.A.S., Honorary Member of the Institute of Actuaries.*

PART III.—*On the methods of Computation and Verification of the Table.*

THE twelve-figure table which accompanied Part I. was extracted from a table extending to twenty-four places, which is still in manuscript. Hence the details I am about to give of the methods of construction and verification made use of, will necessarily have reference to the more extensive table.

The table comprises five* columns, related to each other as in the smaller table; and the logarithms, as stated, extend to twenty-

---

* A definite relation subsists between t, the number of places in the tabulated logarithms, m, the number by which the process is designated, and c, the number of columns which it is necessary to exhibit. The number of tabular entries requisite, irrespective of the auxiliary table, is $t:m$. The first half of these require a column each, while for the second half one additional column suffices—one period of m figures being dropped at the second entry in it, two at the third, and so on. Hence, generally,

$$c=\frac{t}{2m}+1.$$

When $m=3$ this gives, for $t=12$, $c=3$; and for $t=24$, $c=5$.

The property of which advantage is thus taken to restrict the number of columns is an obvious consequence of the relation

$$\log(1+n)=\text{M}(n-\frac{n^2}{2}+\frac{n^3}{3}-\ldots),$$

the first significant figure of the greatest value of n, in Col. V., being in the thirteenth decimal place.

four places, the preliminary periods of ciphers, in the second and following columns, being suppressed, as in the table referred to. The logarithms were all formed true to twenty-seven places, to enable the twenty-fourth to be made true to the nearest unit. The last four columns were constructed by the method of differences; but the first column, which does not admit of the advantageous application of that method, had to be formed otherwise. I shall first describe the formation of Col. I.

The logarithms in Col. I., indices being neglected, are those of 1000 to 1999; and could I have found them, to the extent wanted, in any table of repute, I should not have scrupled to appropriate them. But I know of no such table containing them. The materials within my reach, and which sufficed for the purpose, were Abraham Sharpe's Table of Common Logarithms to sixty-one places. and Wolfram's Table of Naperian Logarithms to forty-eight places.

Sharpe's table is contained in the earlier editions of Hutton, and also in Callet. It is wanting in my copy of Hutton, and I used it as I found it in Callet. This table gives the logarithms of all numbers to 100, and of the primes up to 1,100. From it therefore I got, directly, the logarithms of the primes from 1,000 to 1,100; and also, by addition of their components, the logarithms of all the composites comprised between my limits. In most cases I took the first fifteen figures of the logarithms abstracted, from Hutton's table to twenty places, on account of the superior legibility of this table to that of the corresponding one of Callet.

Wolfram's table is contained in *Vega's Tables*[*] (Leipzig, 1794, folio). It gives the Naperian logarithms (to forty-eight places) of all numbers from 1 to 2,200, and of the primes up to 10,000. By its aid I formed the logarithms of the remaining primes, 119 in number. To obtain the common logarithms from those given by Wolfram, multiplication by the modulus ·43429 . . . was necessary in each case. This operation was much facilitated by the employment of a table, which I formed on purpose, of the first thousand integer multiples of M to thirty-three places.

The one thousand logarithms composing Col. I. being thus formed, there remained the necessity for their verification. For, whatever amount of care I might have exercised—and I had spared no pains to secure accuracy—it would have been too much to expect that in such a number of extensive and independent operations no error should have escaped notice; and besides, I did not know what reliance was to be placed upon the tables I had made use of. Now, there is a theorem which enables us to assign the sum of any number of consecutive logarithms, and I therefore determined, using this theorem, to verify the logarithms I had formed by addition.

---

[*] This work is very valuable, and is also, I believe, very scarce. The copy I possess belonged to Mr. Baily, at the sale of whose library it was bought by Mr. Woollgar, who bequeathed it to me.

Having inserted the logarithms in the column they were to occupy (writing the last nine figures—the nineteenth to the twenty-seventh—of each in pencil, so as to admit of easy correction and curtailment), I added them in groups of ten, and the sums of these again in groups of five, and thus obtained the sums of the successive fifties, ending with 1,049, 1,099, 1,149, 1,199, &c. I then, by the application of the theorem above referred to, determined the sums of the same fifties as follows:—

The theorem is known as "Stirling's Theorem," having been first given by that mathematician in his *Methodus Differentialis;* but it is now to be met with in most treatises on the Differential Calculus and the Calculus of Differences, where it is given as a particular application of a theorem of much greater generality.* As adapted to common logarithms, the theorem is,

$$\log(1.2.3\ldots\overline{x-1}),\ \text{or}\ \log 1+\log 2+\log 3+\ldots+\log(x-1)$$
$$=\log\sqrt{(2\pi)}+(x-\tfrac{1}{2})\log x-\mathrm{M}\left(x-\frac{\mathrm{B}_1}{1.2x}+\frac{\mathrm{B}_3}{3.4x^3}-\frac{\mathrm{B}_5}{5.6x^5}+\ldots\right),$$

where M is, as usual, the modulus, and B_1, B_3, &c., are the *Numbers of Bernoulli*. These Numbers being tabulated, we have the means of carrying the series that multiplies M to any extent that may be requisite. This depends on the values given to x, and on the number of places that are required to be true in the results. For values of x from 1,000 to 2,000, and thirty places true in the results, the following suffices, on putting in the values of B_1, B_3, &c.:—

$$\log\sqrt{(2\pi)}+(x-\tfrac{1}{2})\log x-\mathrm{M}\left(x-\frac{1}{12x}+\frac{1}{360x^3}-\frac{1}{1260x^5}+\frac{1}{1680x^7}\right).$$

In this form and to this extent I applied the theorem between the limits† 1,000, 1,050, 1,100, 1,150, &c., up to 2,000; and the results ought to have agreed with the sums previously obtained by actual addition. In some cases they did so agree, but in most they did not, and I had much trouble in tracing the various discrepancies to their source. I succeeded in removing them all however, and brought the two sets of results into entire accordance as far as the twenty-sixth decimal place. Some of the discrepancies I found arose from arithmetical errors committed by myself, and the others originated in errors in the tables I had employed. Of these tabular errors, I discovered no fewer than thirteen, namely, seven in Hutton, five in Callet, and one in Wolfram. I give a list of them below.*

---

* See in particular De Morgan's *Differential and Integral Calculus*, p. 312, and Boole's *Calculus of Finite Differences*, pp. 86, 87.

† It is hardly necessary to mention, that in applying the theorem between limits, the constant, $\log\sqrt{(2\pi)}$, disappears.

But although the correctness of the *sums* of the successive fifties was thus sufficiently established, it by no means followed that the individual values composing them were correct; there might be compensating errors, which computers know are of not particularly rare occurrence. Such errors, therefore, had still to be sought for and eliminated. The process I employed for this purpose (which I need not more particularly describe) was equivalent to the differencing out of the whole column, as far as the differences of the seventh order. In an order of differences so remote, any error in the numbers operated upon would show itself in an intensely aggravated form. In the application of this process no errors were discovered; and hence the accuracy of Col. I. may be considered as satisfactorily established.

The remaining columns were constructed by the method of differences. Fundamental values were obtained by the formation of each hundredth term by the theorem,

$$\log(1+n)=\mathrm{M}\left(n-\frac{n^2}{2}+\frac{n^3}{3}-\ldots\ldots\right).$$

By a first interpolation, nine terms were inserted in each interval of the series thus formed; and by a second, nine terms were again inserted in each interval of the new series. The number of orders of differences it was necessary to make use of in the several interpolations were, in Col. II., seven and five; in Col. III., four and three; in Col. IV., three and two; and in Col. V., two and one.

---

* The tables in which the errors were found are Hutton's and Callet's tables to twenty places, and Wolfram's table to forty-eight places. The logarithms being in all the tables arranged in periods, it is sufficient to give the erroneous period and its correction.

Hutton.

log 149,	~~1~~3826	should be	03826
1071,	94~~6~~08	„	94708
1085,	8~~51~~48	„	84548
1105,	21~~7~~29	„	21129
1115,	84~~7~~79	„	84179
1125,	47~~9~~81	„	47381
1135,	29~~7~~41	„	29141

Callet.

log 965,	5~~8~~538	should be	56538
1022,	9~~0~~700	„	92700
1082,	7260~~8~~	„	72607
1154,	5808~~7~~	„	58088
1158,	8559~~6~~	„	85593

Wolfram.

log 1409, 16~~8~~6 should be 1696.

As the whole of the above errors may not be found in all the editions of Hutton and Callet, it is proper that I should mention that my copies of these works are of the editions following:—Hutton, 7th edition, 1838; and Callet, "(1795) An 3^{e}." I find also that the two twenty-figure tables referred to, are very far from being universally true to the nearest figure in the twentieth place.

Series formed by interpolation, as just described, need no other verification than that which is afforded in the course of the operation. Hence, in the case of Cols. II. to V., it was only necessary to guard against errors of copying. With this view, after the terms composing these columns were inserted in their places, they were added in groups of ten; and the sums being differenced out as far as was necessary, the regularity in the progression of the last order of differences formed, furnished sufficient assurance of the accuracy of the whole.*

The table hereto appended, as already stated, has been extracted from the larger table whose construction has just been described. After careful reading with the copy, the sums of every ten terms were compared with those formed for verification of the larger table; and I trust it will be found that the precautions thus taken to secure accuracy have been successful.

Before leaving the subject of the twenty-four-figure table, I give an example of its application, being one that involves the use of both the direct and the inverse processes.

Given

$$\Delta^2 \log^{-1}(\cdot 1376) = (10^{\cdot 0001} - 1)^2 10^{\cdot 1376};$$

it is required to evaluate the number thus symbolized.

If we call the number required N, we shall have

$$\begin{aligned} \log \mathrm{N} &= 2 \log(10^{\cdot 0001} - 1) + \log 10^{\cdot 1376} \\ &= 2 \log(10^{\cdot 0001} - 1) + \cdot 1376. \end{aligned}$$

By the exponential theorem,†

$$a^x - 1 = \mathrm{A}x + \frac{\mathrm{A}^2 x^2}{1.2} + \frac{\mathrm{A}^3 x^3}{1.2.3} + \frac{\mathrm{A}^4 x^4}{1.2.3.4} + \ldots\ldots,$$

where A is the Naperian logarithm of a. Making in this $a = 10$, A consequently $= 2\cdot 302585\ldots$, and $x = \cdot 0001$, we find

$$10^{\cdot 0001} - 1 = \cdot 00023{,}02850{,}20824{,}75268{,}35942{,}55671{,}94133\ldots.$$

---

* I carefully preserve these sums, as well as the corresponding sums of Col. I., for use in verifying the several columns, in case the large table should be hereafter put in type.

† There is another theorem that would seem more directly to answer the end in view, namely,

$$(a^x - 1)^n = (\mathrm{A}x)^n + \frac{\Delta^n 0^{n+1}}{1.2\ldots n+1}(\mathrm{A}x)^{n+1} + \frac{\Delta^n 0^{n+2}}{1.2\ldots n+2}(\mathrm{A}x)^{n+2} + \ldots,$$

which, on making $n = 2$, and taking the values of $\Delta^2 0^3$, $\Delta^2 0^4$, &c., from a table of the *Differences of Nothing*, becomes

$$(a^x - 1)^2 = (\mathrm{A}x)^2 + (\mathrm{A}x)^3 + \frac{7(\mathrm{A}x)^4}{12} + \frac{(\mathrm{A}x)^5}{4} + \frac{31(\mathrm{A}x)^6}{360} + \frac{(\mathrm{A}x)^7}{40} + \frac{127(\mathrm{A}x)^8}{20160} + \ldots$$

But the exponential theorem is rather more easily used; and as it is the logarithm of $(10^{\cdot 0001} - 1)^2$ that is wanted, the theorem last named answers the purpose just as well.

The rest of the operation, involving the use of the tables, is as follows —

$10^{\cdot 0001}-1=$	·0 002 302 850 208 247 526 835 942 557	÷2	
	1·151)425 104 123 763 417 971 278	369	
	79 80		
	10 744		
	1·151 424 719)385 123 763 4	334	
	39 696 347 71		
	5 153 606 147		
	1·151 425 103 575 85$\cancel{0}$)547 907 271 971 278	475	
	87 337 230 540 936		
	6 737 473 290 626		
	1·151 425 104 12$\cancel{3}$)980 347 772 747	851	
	59 207 689 449		
	1 636 434 243		
	485 009 139	421	
	24 439 097		
	1 410 595		
	259 170	225	
	28 885		
	5 856		
	99	086	
	7		
	301 029 995 663 981 195 213 739	log 2	
	61 075 323 629 791 801 848 963	151	Col. 1
	160 225 104 108 290 795 851	369	„ 2
	145 054 332 731 613 893	334	„ 3
	206 289 878 855 051	475	„ 4
	369 584 604 100	851	„ 5
	182 837 976	421	„ 6
	97 716	225	„ 7
	37	086	„ 8
$\log(10^{\cdot 0001}-1)=$	$\bar{4}$·362 265 689 658 873 665 867 326	×2	
„ $(10^{\cdot 0001}-1)^2=$	$\bar{8}$·724 531 379 317 747 331 734 652		
„ $10^{\cdot 1376}$ =	0·137 6		
„ N =	$\bar{8}$·862 131 379 317 747 331 734 652	log 6	
	778 151 250 383 643 632 508 767		
	83 980 128 934 103 699 225 885	213	Col. 1
	83 860 800 866 572 974 202 474		
	119 328 067 530 725 023 411	274	„ 2
	118 980 388 472 549 269 441		
	347 679 058 175 753 970		

347 679 058 175 753 970	800	Col. 3
347 435 446 548 441 373		
243 611 627 312 597	560	„ 4
243 204 909 797 724		
406 717 514 873	936	„ 5
406 499 635 061		
217 879 812	501	„ 6
217 581 535		
298 277	686	„ 7
297 926		
351	808	„ 8
351		

1·213	274
242 6	
84 91	
4 852	
1·213 332 362	800
970 665 889 6	
1·213 333 332 665 88$\cancel{9}$ 6	560
606 666 666 332 945	
72 799 999 959 953	
1·213 333 333 34$\cancel{5}$ 356 266 292 898	936
1 092 000 000 011	
36 400 000 000	
7 280 000 000	501
606 666 667	
1 213 333	686
728 000	
97 066	
7 280	808
971	
10	
1·213 333 333 346 492 555 006 236	×6
N=0·00 000 007 280 000 000 078 955 330 037 416	

I use short division here in the resolving process, and the figures shown are all that it is necessary to set down. I might have used log 7 instead of log 6 in the decomposition of log N. Had I done so, the process would have been shortened, as Cols. 2

and 3 would have been passed over; but I wished to present the process in its fullest extent.

I came upon the above somewhat remarkable number when constructing a table of $\log^{-1}x$; and being unable, with the means immediately available, to get to the end of the array of ciphers that follow the first three significant figures, I was startled by its seemingly near approach to rationality.

I take this opportunity of presenting a variety of the compounding process, as applied to the above example.

1·000 000 000 000 936 501 686 80$\cancel{8}$	213
200 000 000 000 187 300 337 362	
10 000 000 000 009 365 016 868	
3 000 000 000 002 809 505 060	
1·213 000 000 001 135 976 54$\cancel{8}$ 098	274
242 600 000 000 227 195 309	
84 910 000 000 079 518 358	
4 852 000 000 004 543 906	
1·213 332 362 001 136 28$\cancel{7}$ 803 671	800
970 665 889 600 909 030	
1·213 333 332 667 02$\cancel{8}$ 888 712 701	560
606 666 666 333 513	
72 799 999 960 021	
1·213 333 333 346 492 555 006 235	×6
·00 000 007 280 000 000 078 955 330 037 410	

If we were to form, in the usual way, the product of the last four factors (half the total number), we should find that, owing to their peculiar form, it would be a number of the same form, the last half of the decimal portion of which would consist of the significant figures of the several factors in order. We, therefore, in accordance with this remark, form the product of the last four factors by inspection, and continue the compounding process from this point in the usual way. The advantage we should anticipate from this summary mode of using up half the factors is, however, visibly not realized. There are a hundred figures more written in this last form of the operation than in the other. Many of them, it is true, are ciphers, and *might* be omitted; but this would hardly be safe, and on the whole I much prefer the other form, always supposing that paper ruled in squares is used.

In another and concluding paper I shall give a history of the method.